Progressive Multigrid Eigensolvers for Multiscale Spectral Segmentation

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Spectral Segmentation

Build a weighted graph G = (V, E, W) from the image

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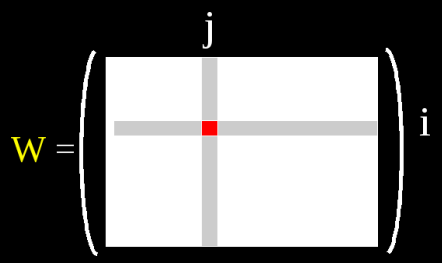
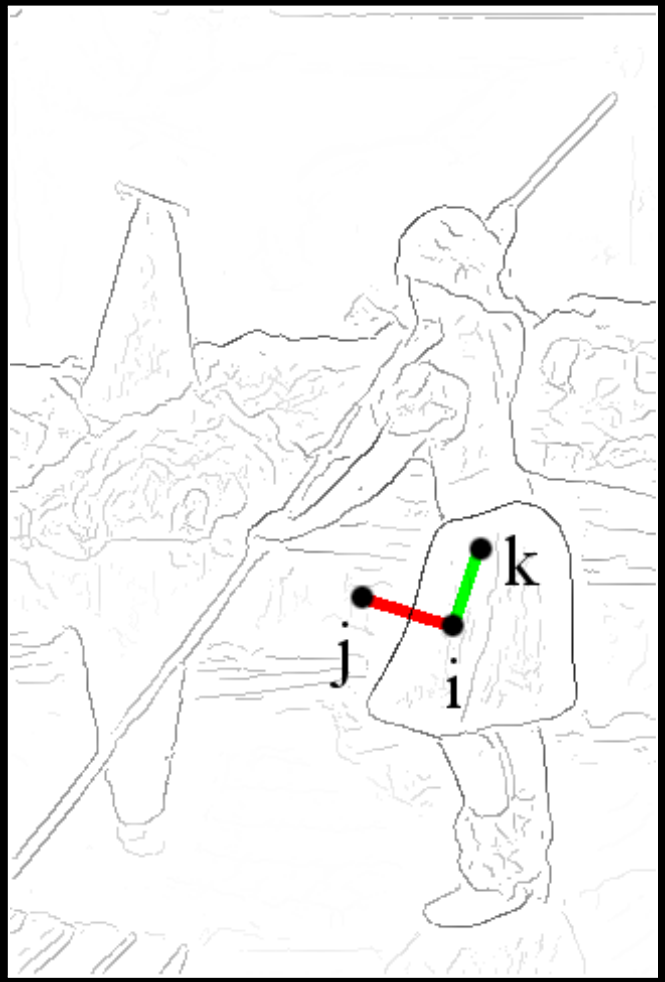


(i, j) low affinity



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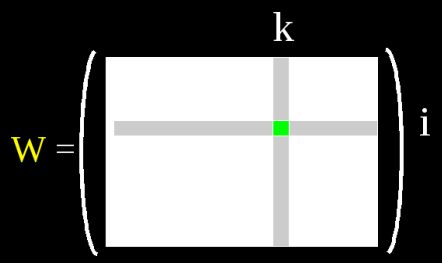

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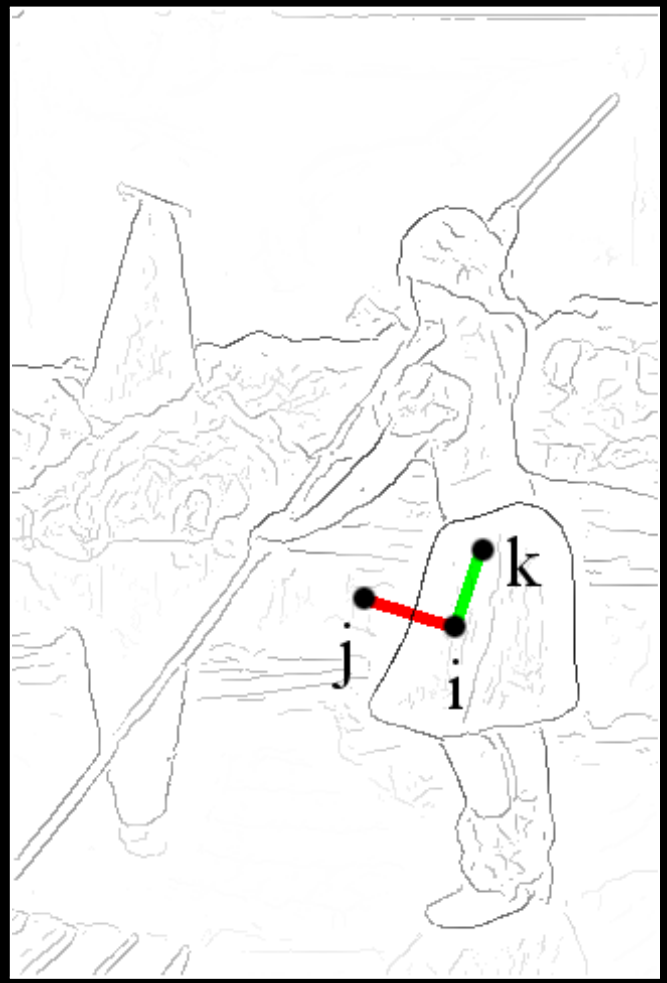

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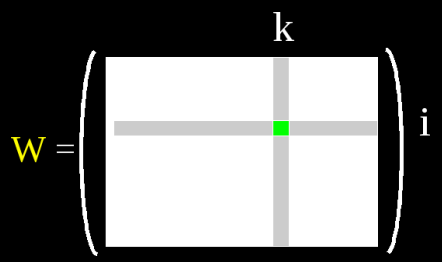

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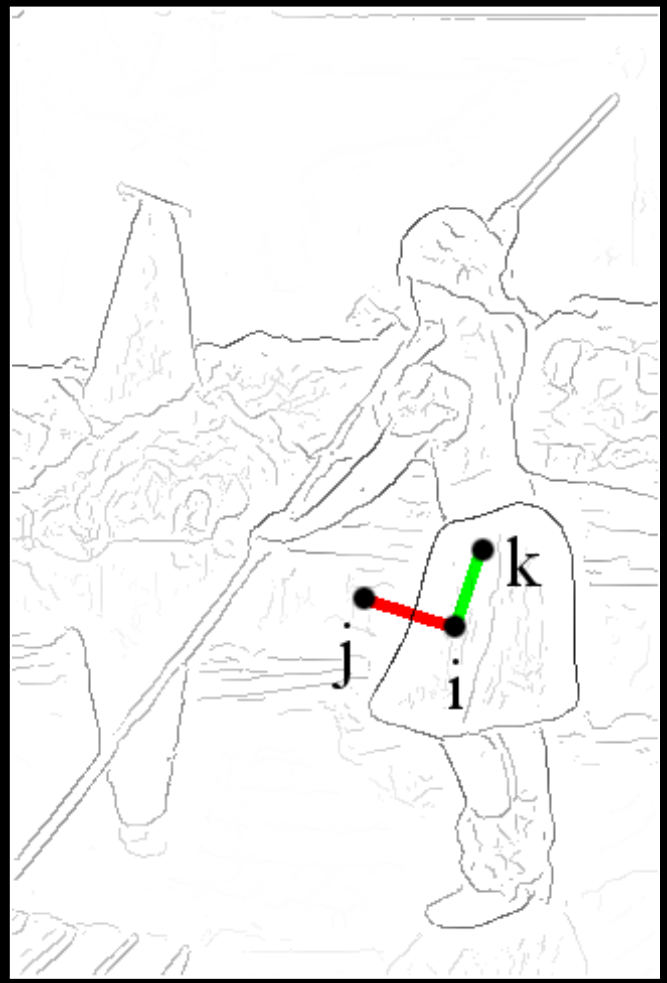
- Normalized Cuts
  [Shi & Malik 1997]



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- General case: partition using smallest eigenvectors of

$$(D - W)z = \lambda Dz$$

where $D_{ii} = \sum_j W_{ij}$

Normalized Cuts: Embedding

Image

Eigenvector

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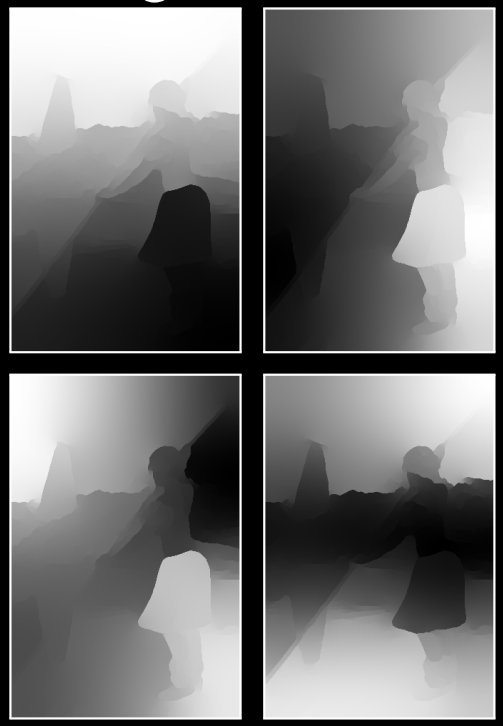
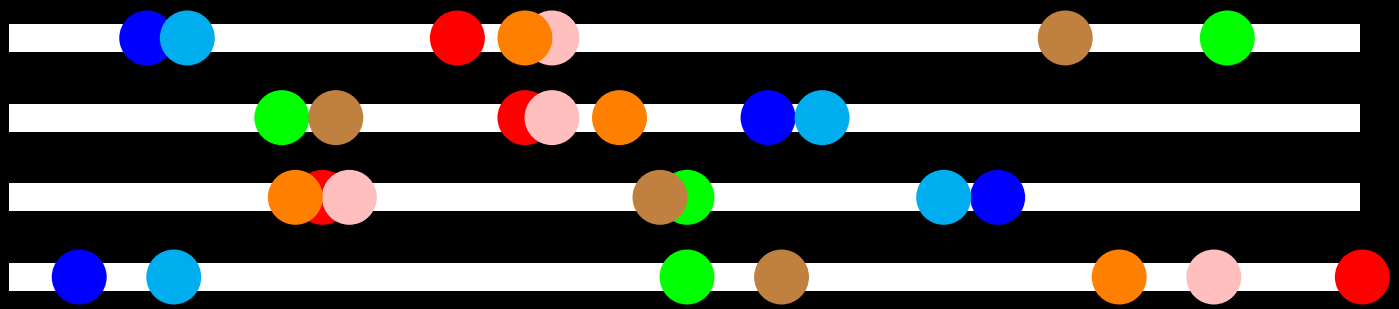
Eigenvectors





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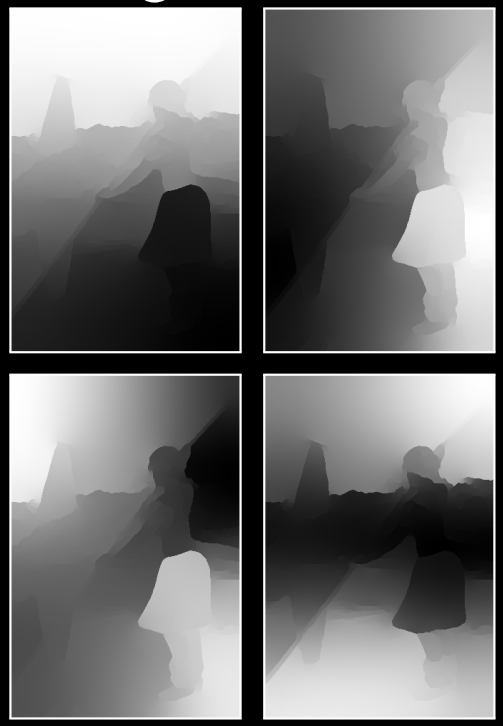
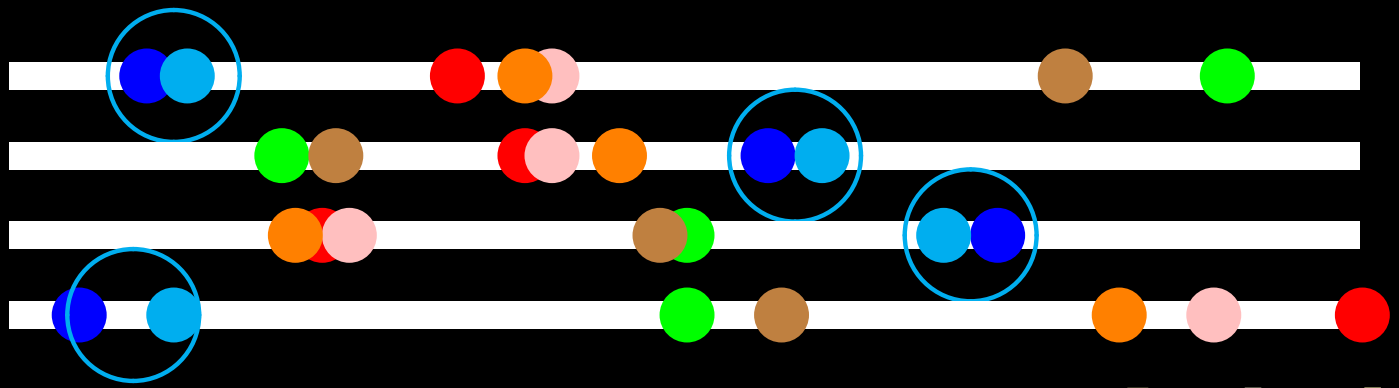
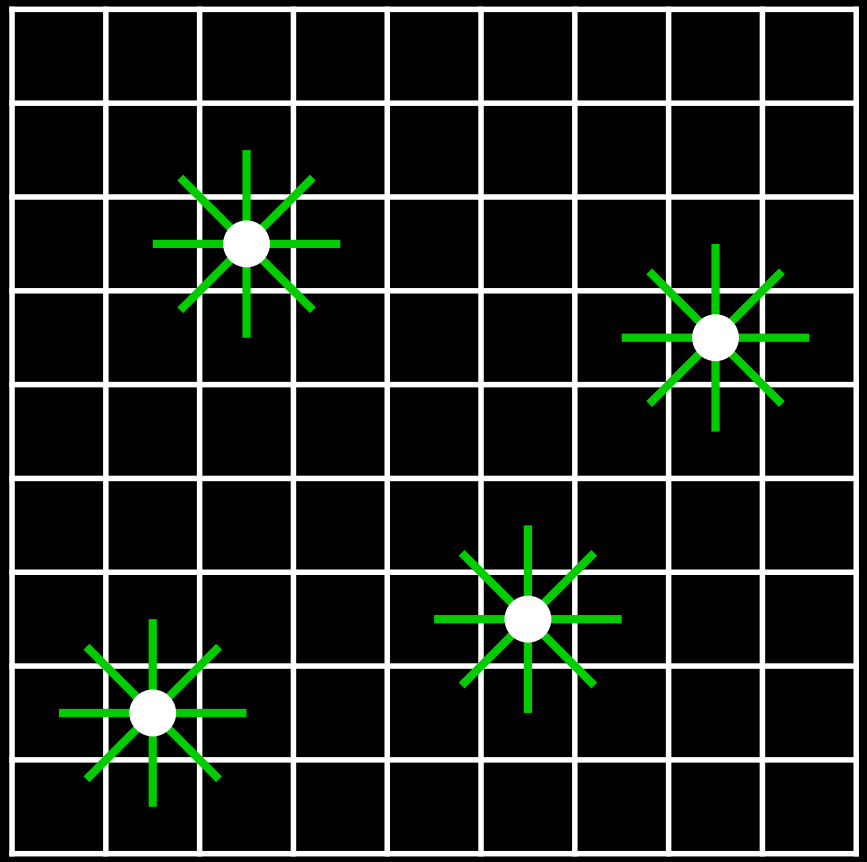


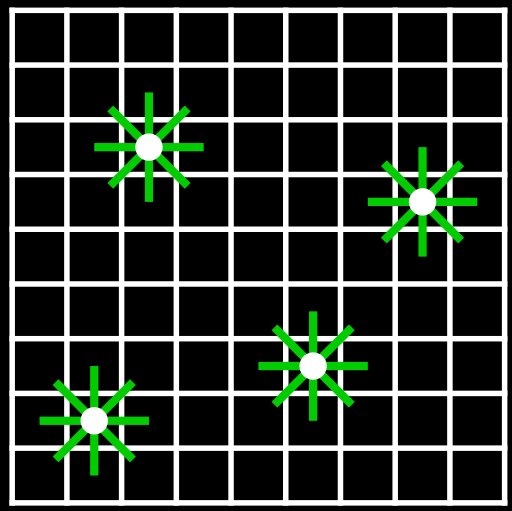


Single Scale Unconstrained Problem



W

Multigrid Solver



W

[Chennubhotla and Jepson, NIPS 2005]
[Brezina et al., Num. Linear Alg. w/App., 2008]
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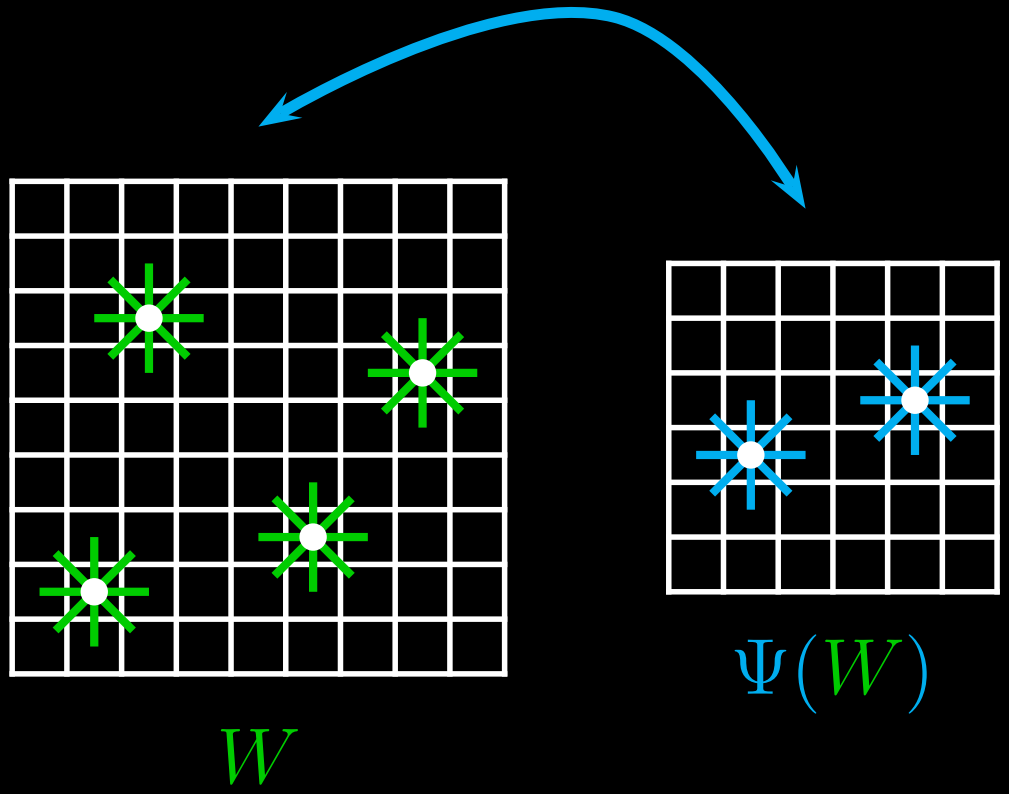

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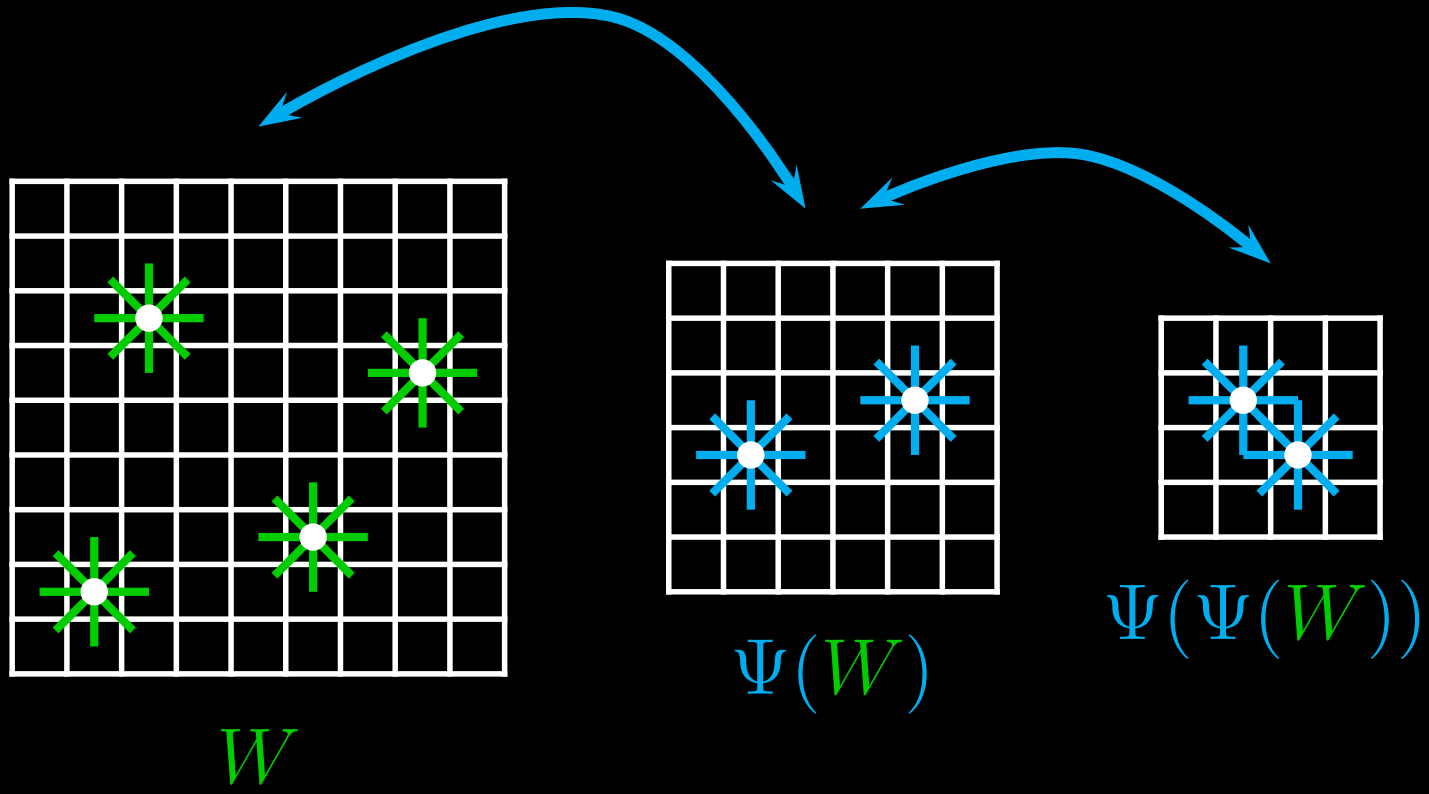

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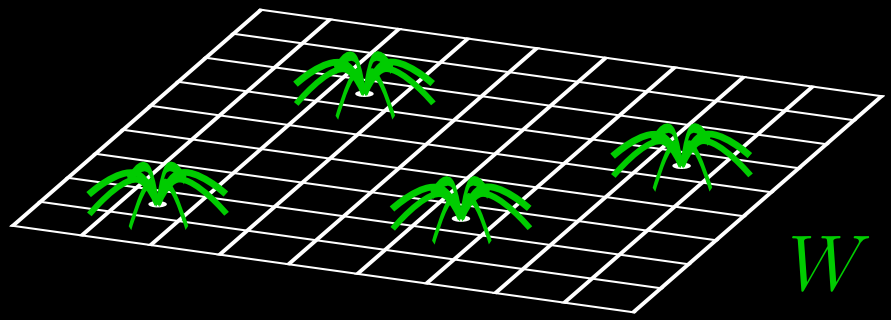
Multirange Problem



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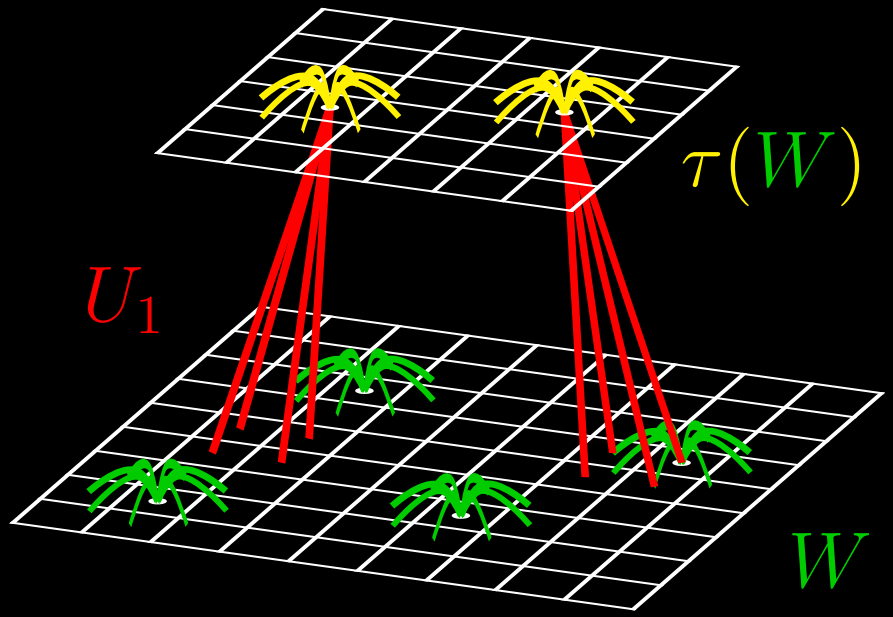

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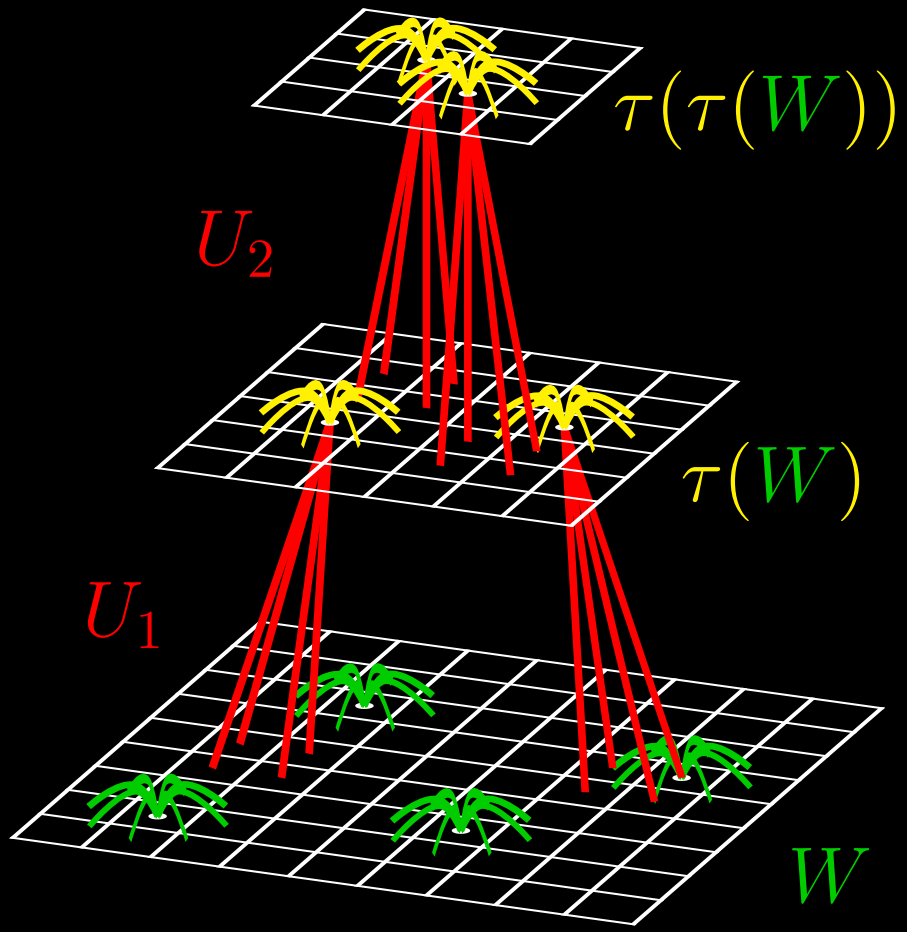

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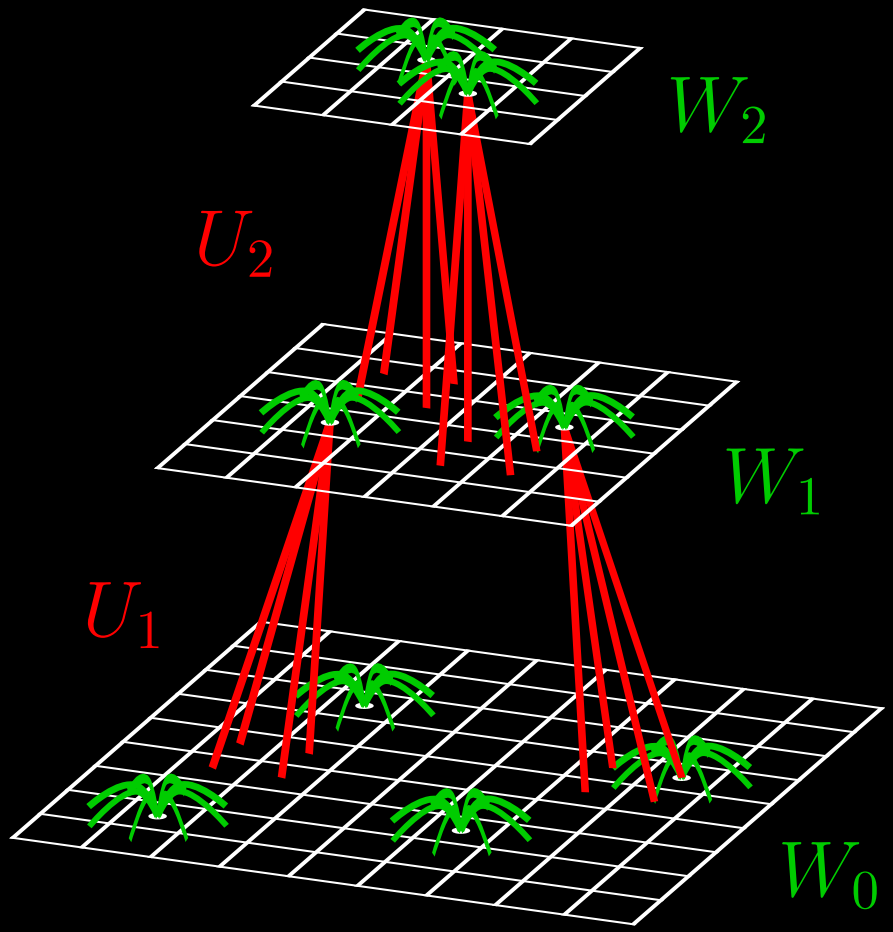
Multiscale Problem



Multigrid and Multiscale

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multigrid: efficient solver computation

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use both!

Multigrid and Multiscale

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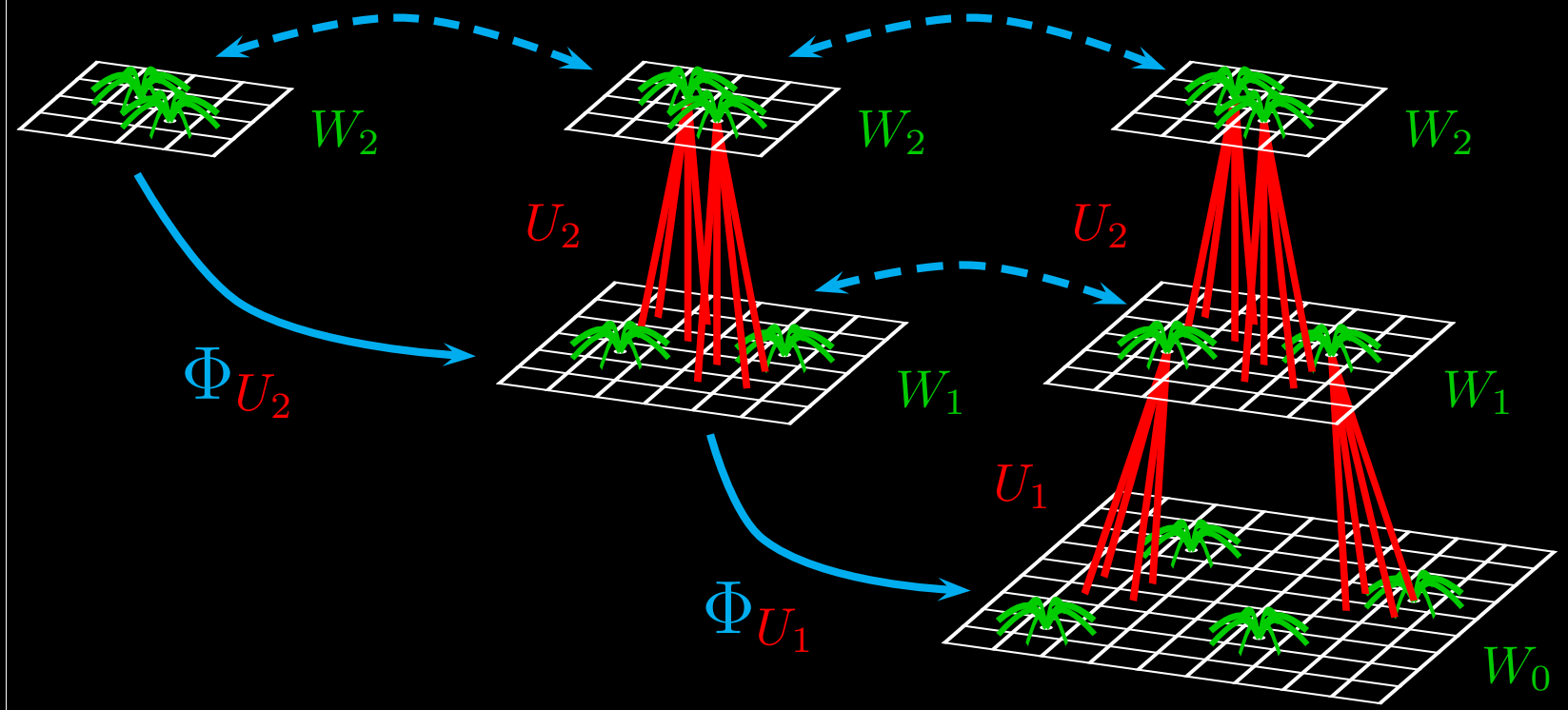
multiscale structure shapes multigrid strategy

Progressive Multigrid Multiscale



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$$W = \begin{bmatrix} W_2 & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & W_0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & U_2 & U_1 \end{bmatrix} \quad Z = \begin{bmatrix} Z_2 \\ Z_1 \\ Z_0 \end{bmatrix}$$

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- Use $[\bar{Z}_2; \; \widetilde{Z}_1]$ as initial guess for solving:

  $(\mathrm{Diag}(W_2, W_1), [0 \;\; U_2])$ for $[\bar{Z}_2; \; \bar{Z}_1]$

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  [Halko, Martinsson, and Tropp, SIREV, 2011]
  - Randomized algorithms for linear algebra problems
  - Exponentially small failure probability
  - Simple implementation
  - Same computational complexity
  - Better hardware parallelization properties

Randomized Matrix Approximation

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- **Fixed Rank Problem**

  Given: $n \times n$ sparse matrix M

  Find: $n \times l$ dense matrix A, where $l = 2m \ll n$

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  - $Y \leftarrow (MM^*)^q M\Omega$
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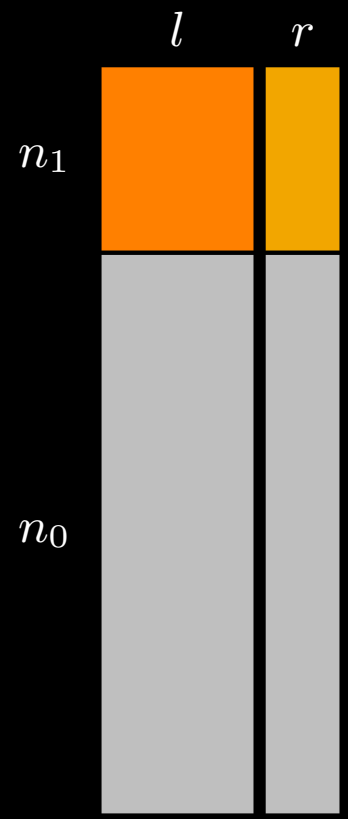
- **Eigensolver**
  - $B \leftarrow A^*MA$ — $l \times l$ matrix
  - $(V, \Lambda) \leftarrow \mathrm{Eigs}(B, m)$ — small eigenproblem
  - $Z \leftarrow AV$

Matrix Approximation + Interpolation

$$M = QPQ$$



Initialize A

Matrix Approximation + Interpolation

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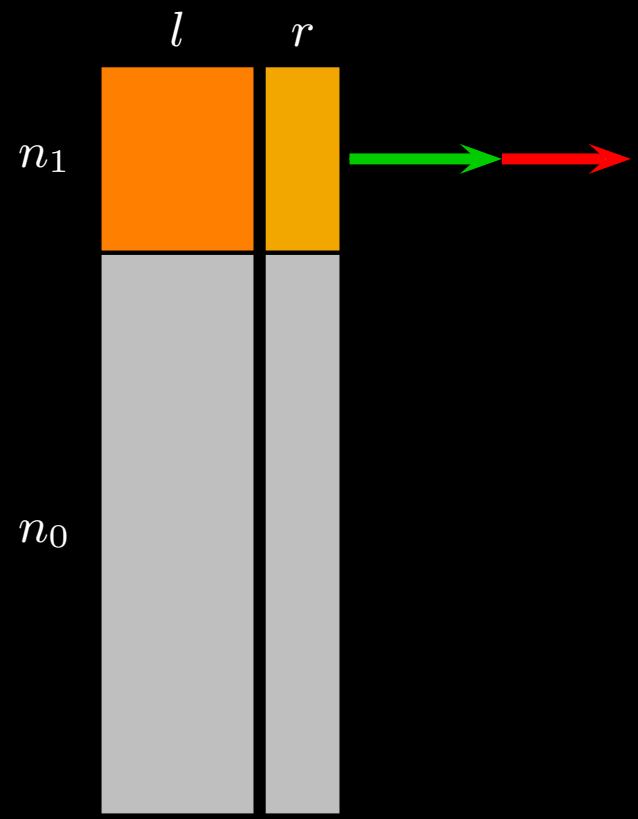

Apply M (Coarse)

Matrix Approximation + Interpolation

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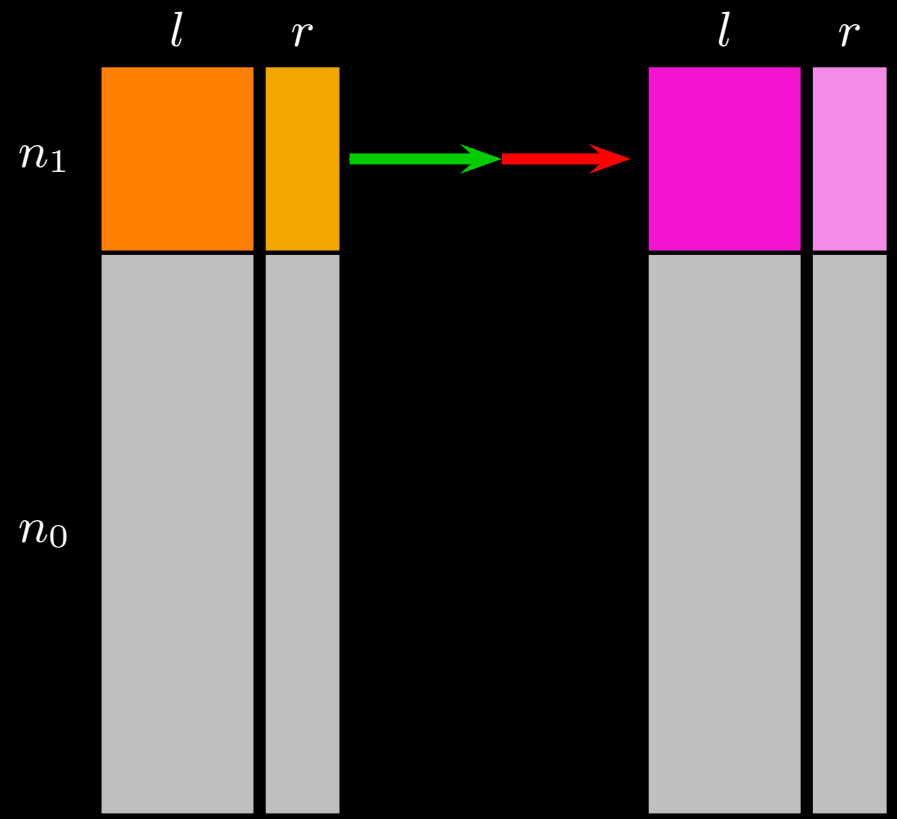

Converge A (Coarse)

Matrix Approximation + Interpolation

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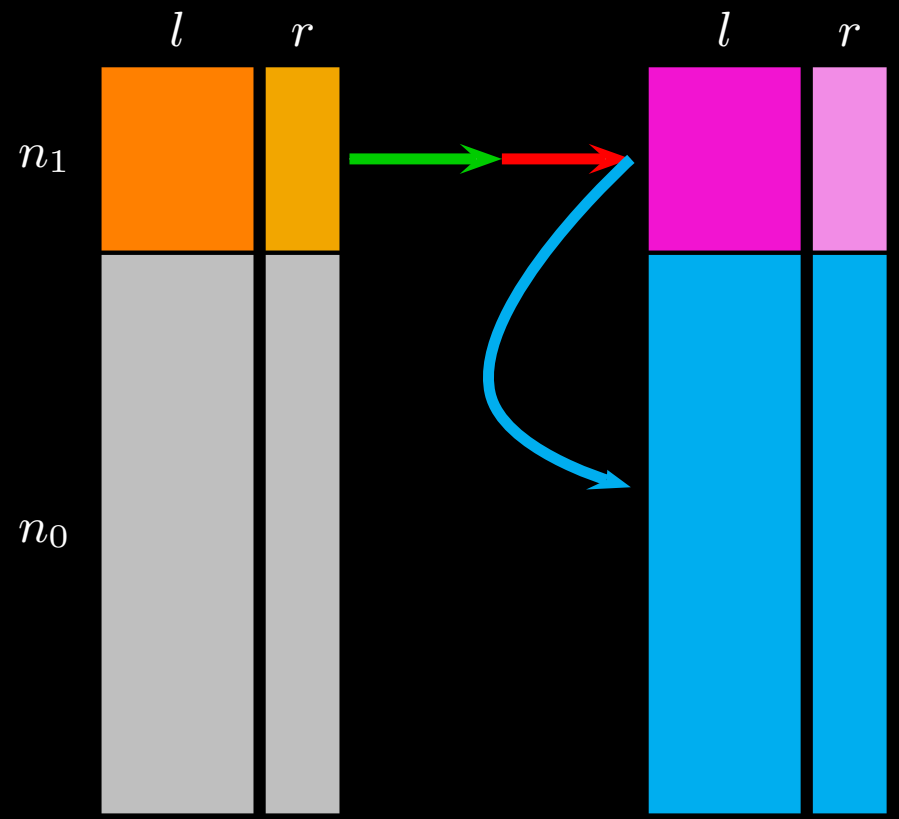

Interpolate

Matrix Approximation + Interpolation

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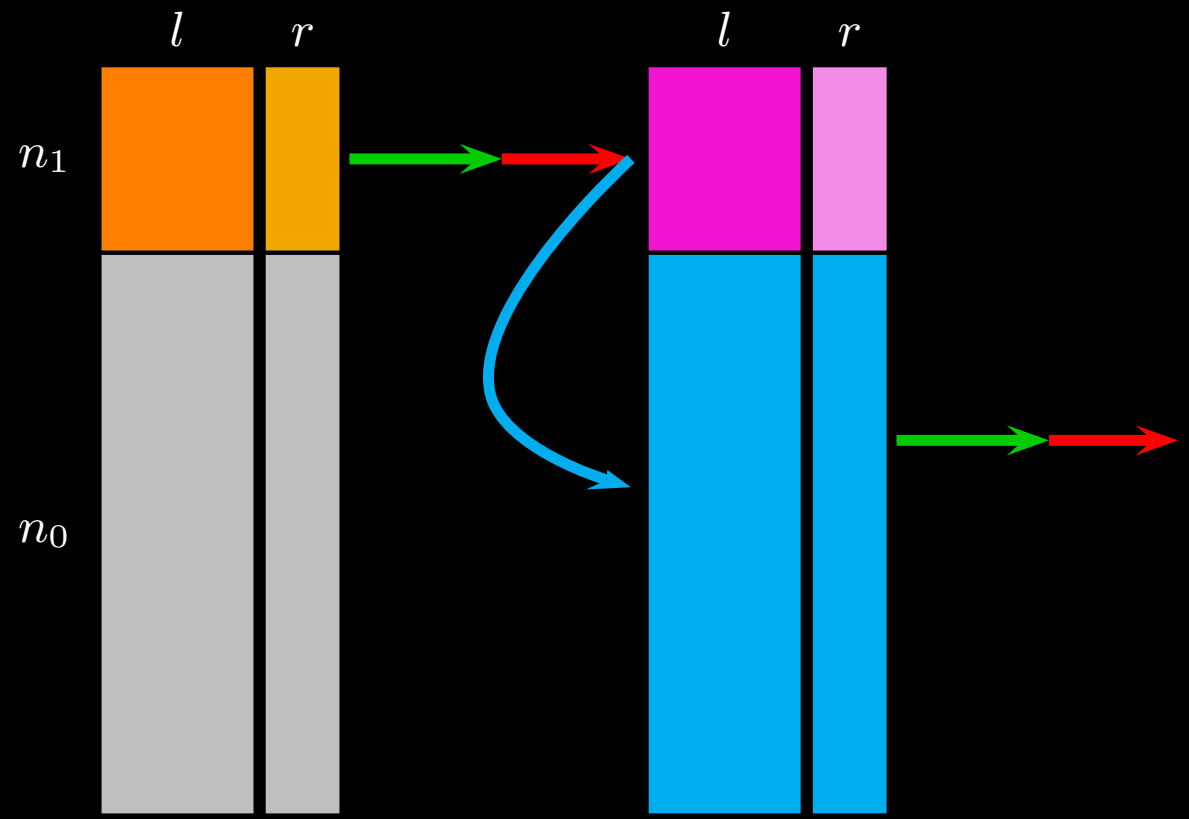
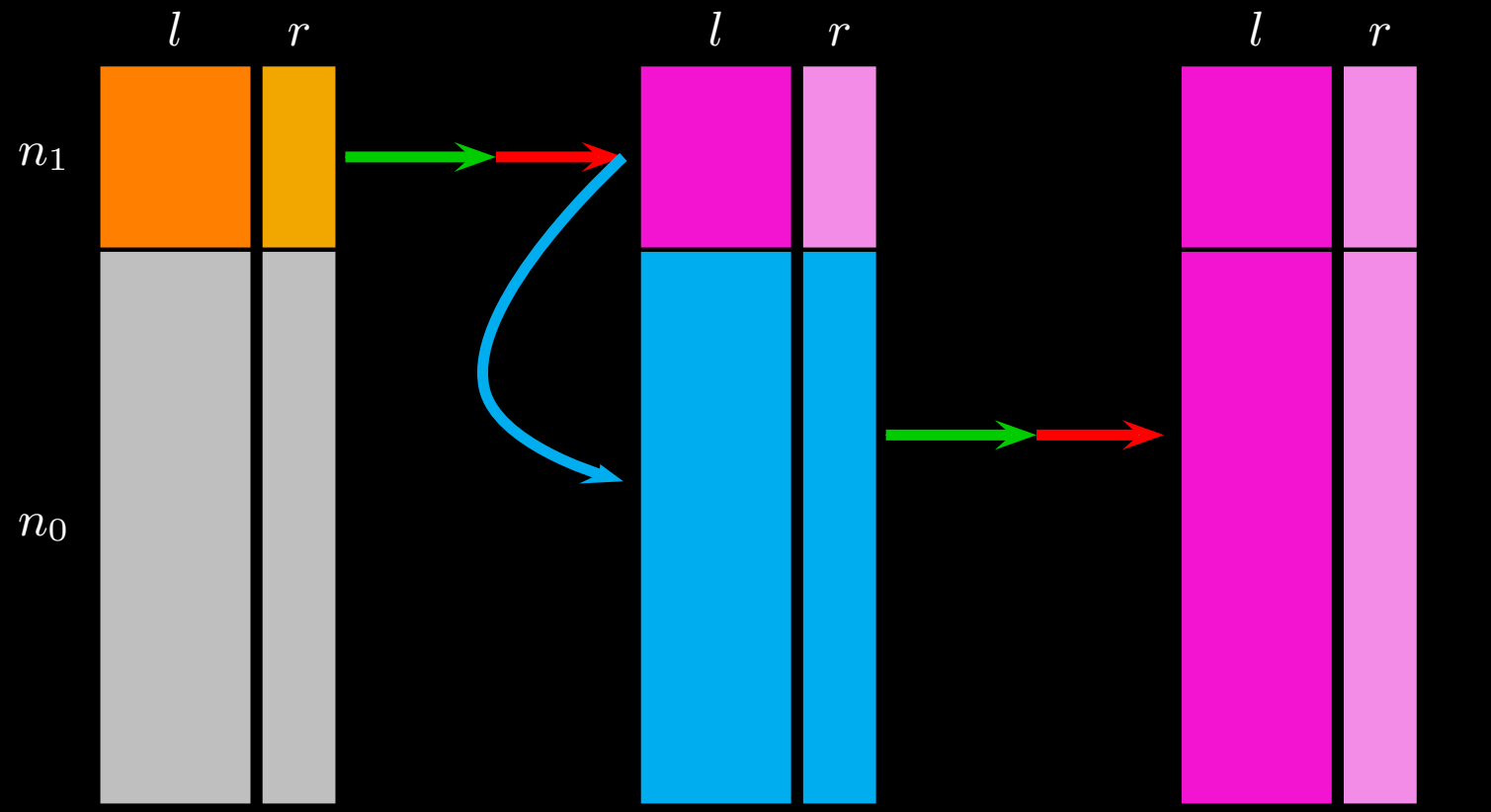


Apply M (Fine)

Matrix Approximation + Interpolation

$$M = QPQ$$



Converge A

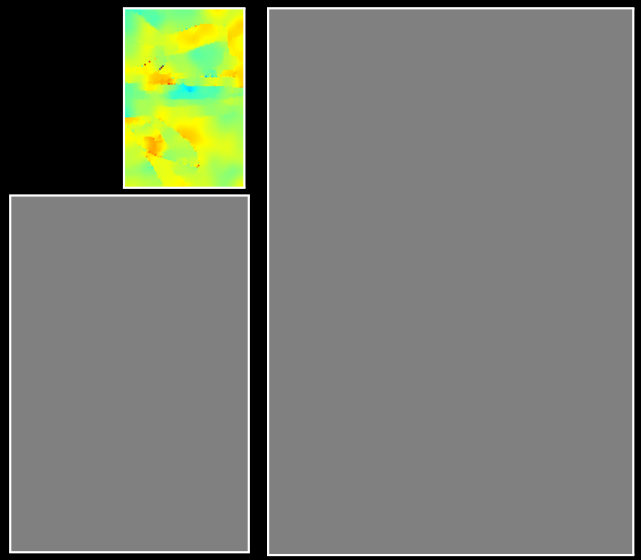
Eigenvector Convergence Comparison

Image

Eigenvector 7

Eigenvector Convergence Comparison

1 coarse: 0.46 sec



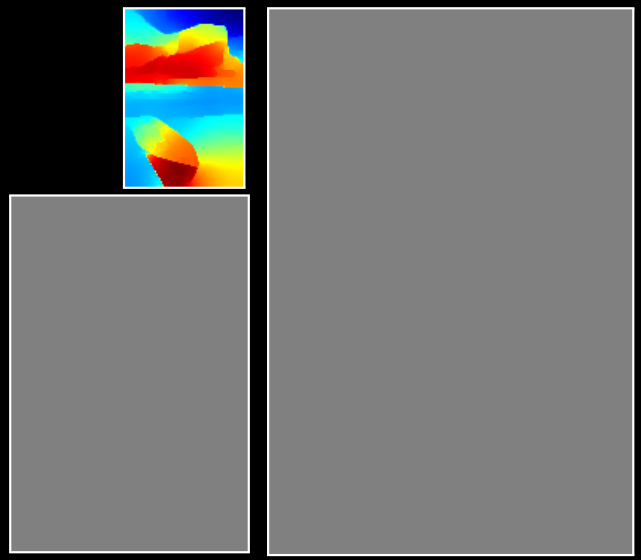
Image

Progressive Multigrid

Eigenvector 7

Eigenvector Convergence Comparison

20 coarse: 3 sec



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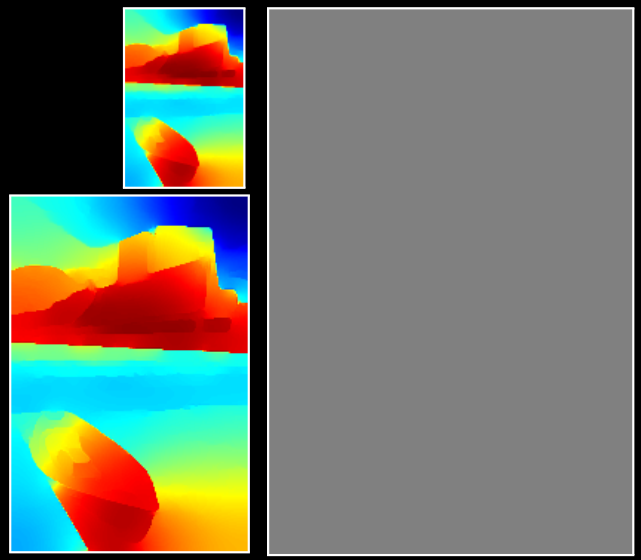
20 c, 1 med: 5 sec

Image



Progressive Multigrid

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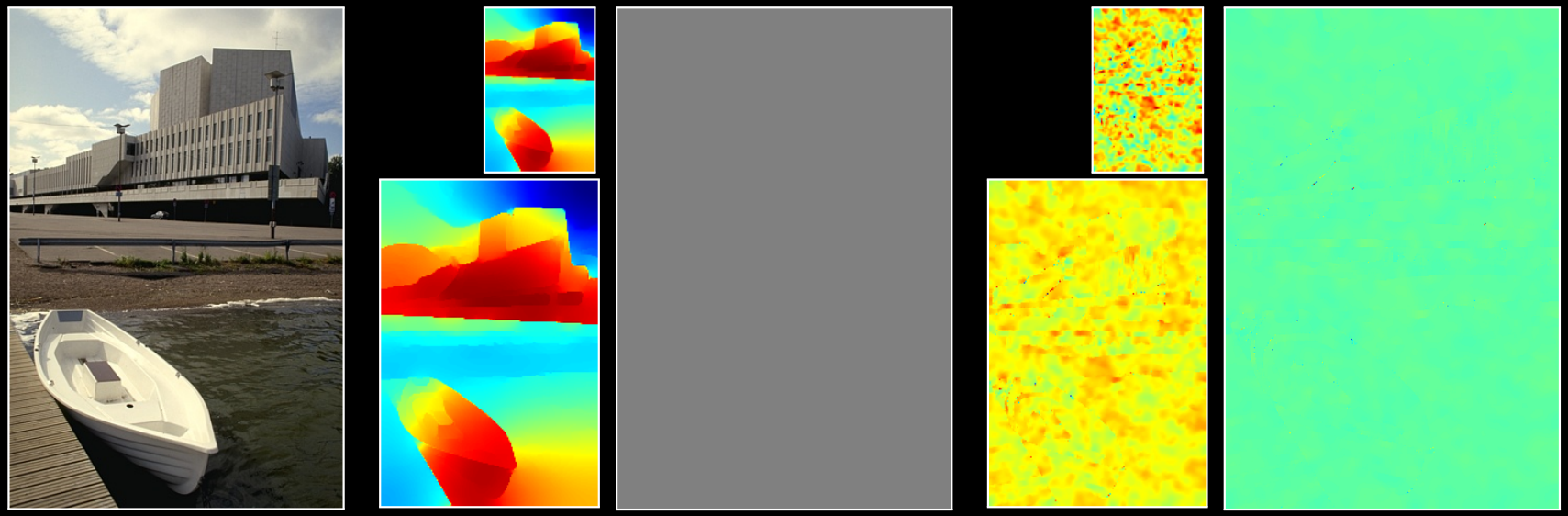

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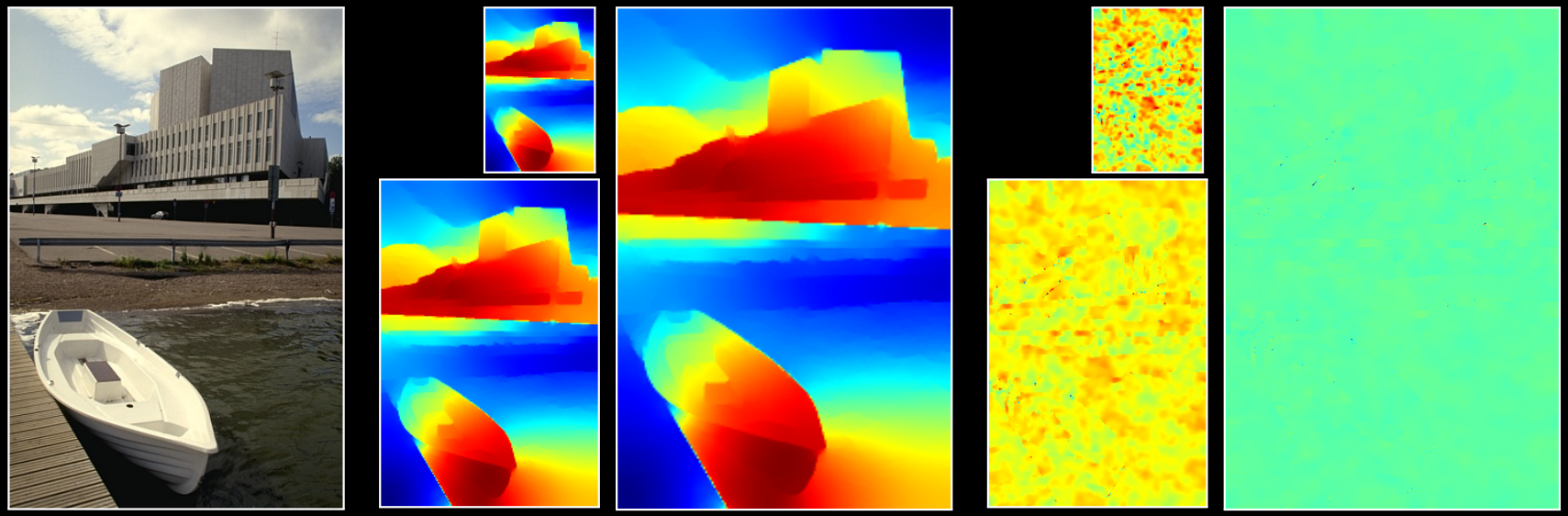

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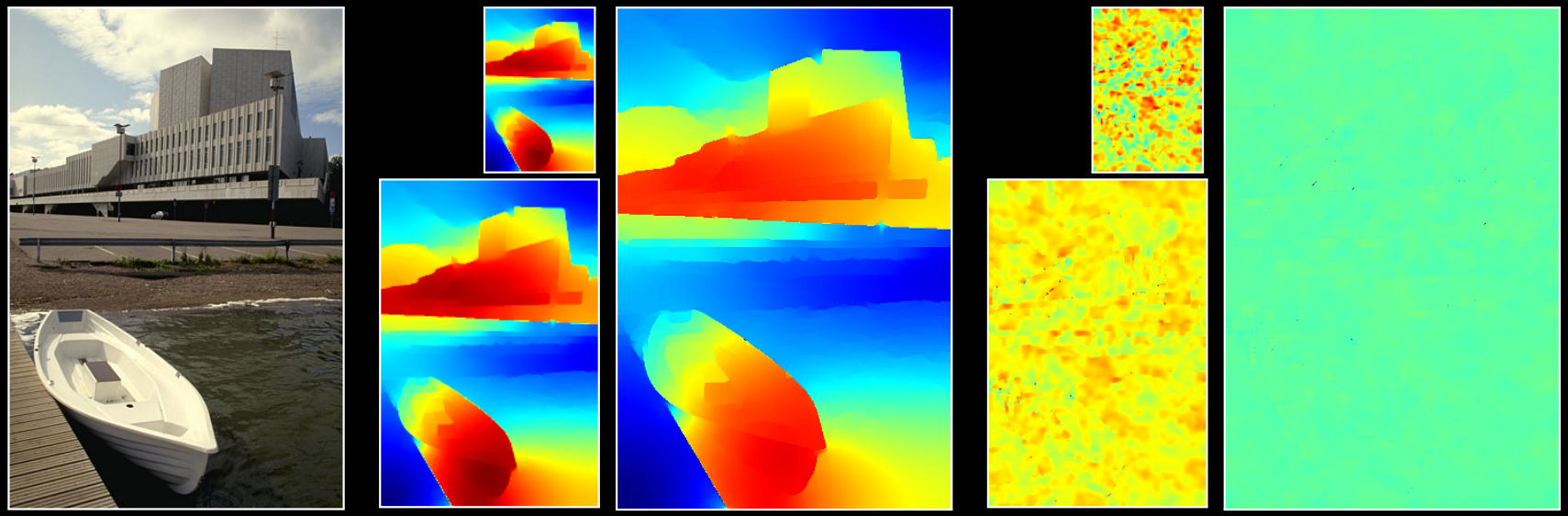

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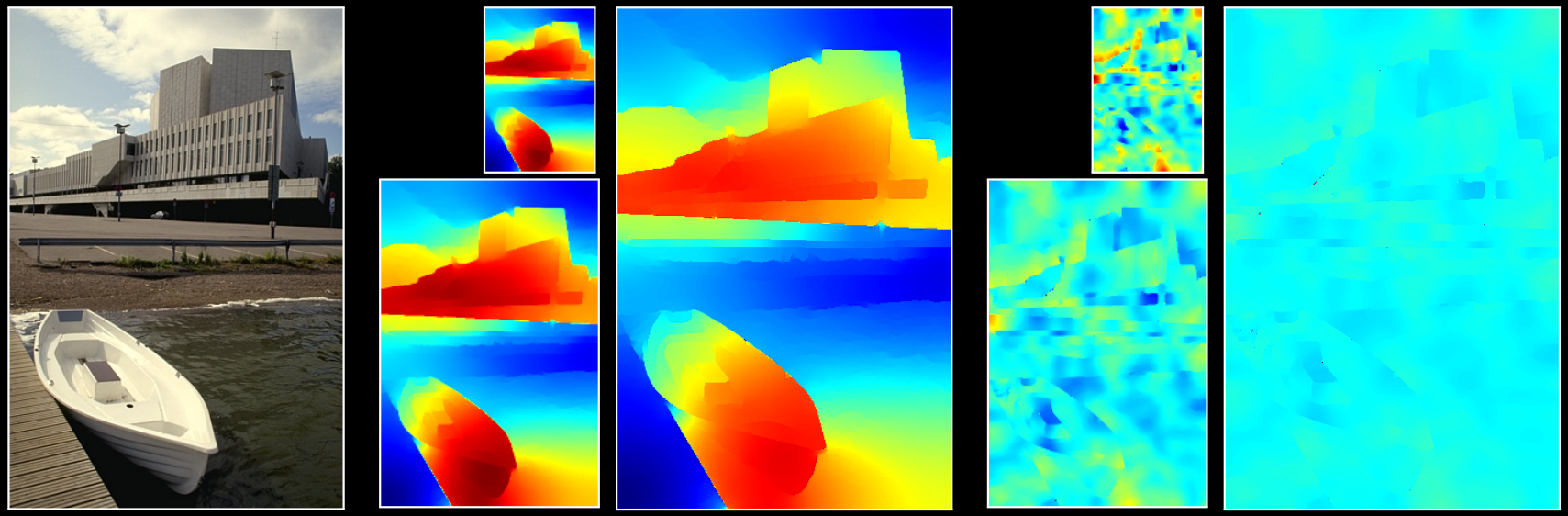

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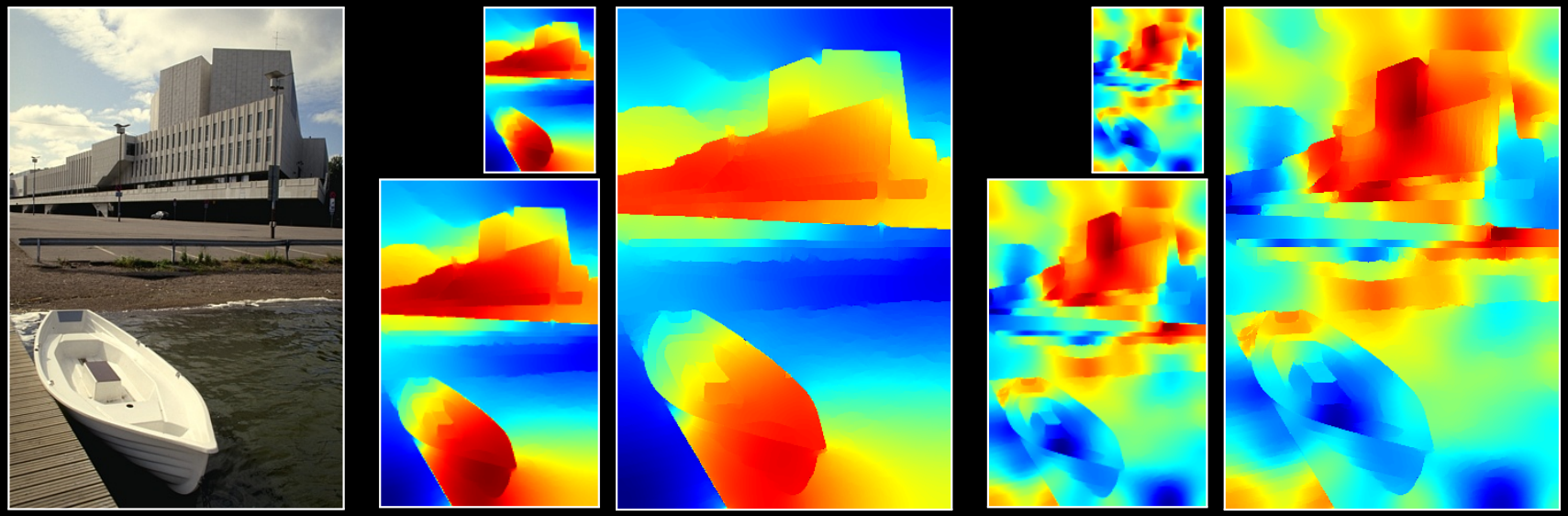

Image | Progressive Multigrid | Baseline Solver

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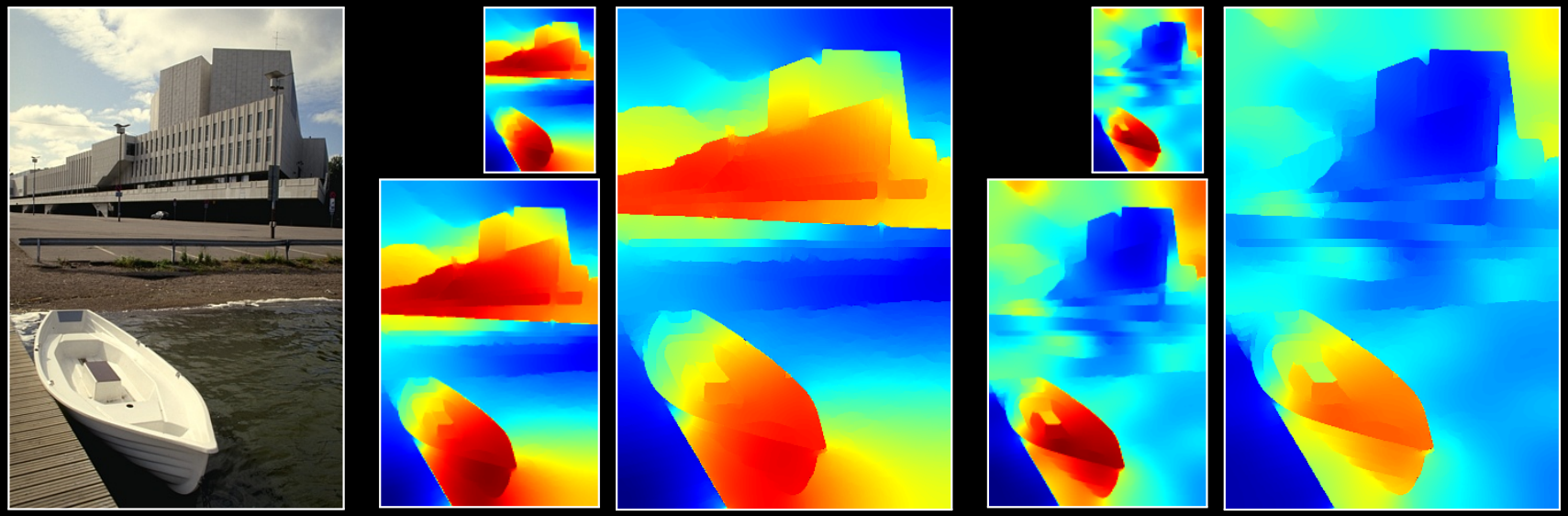

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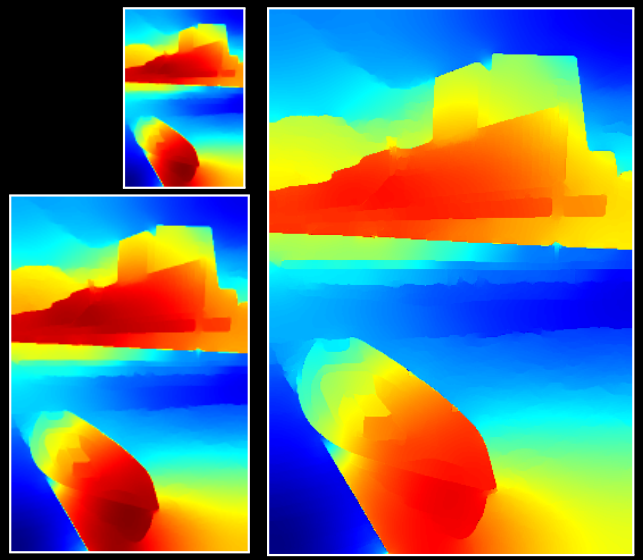
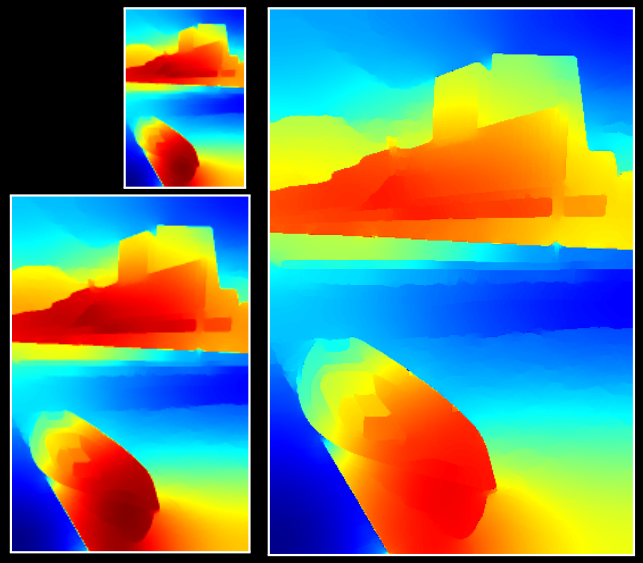
. . . , 3 fine: **27 sec**



Progressive Multigrid

225 fine: **760 sec**



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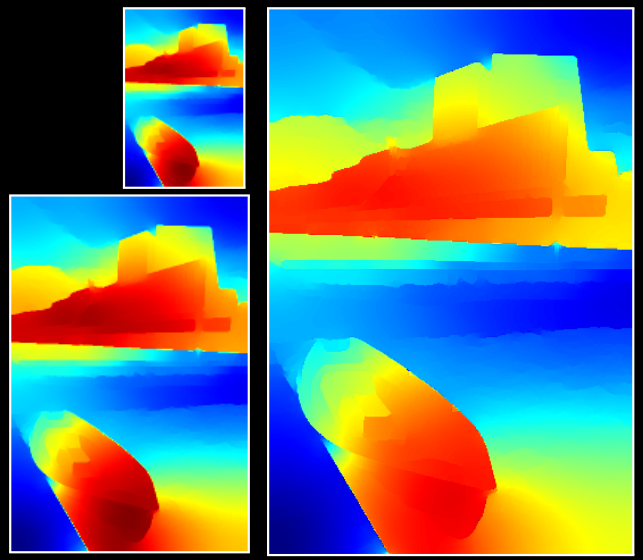
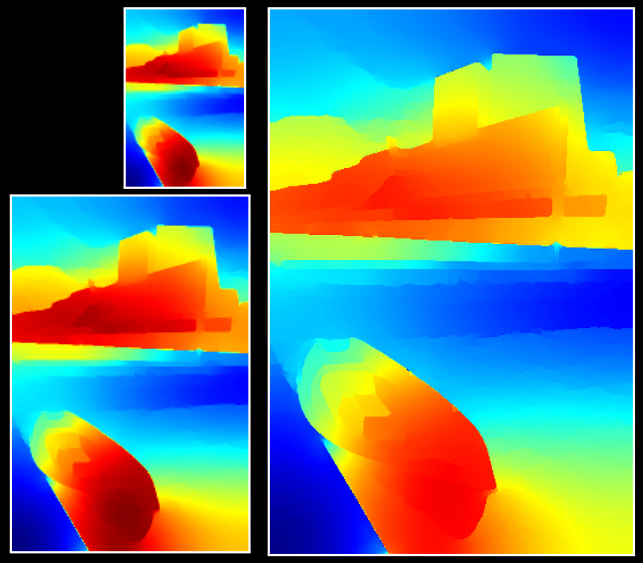


Progressive Multigrid

225 fine: **760 sec**



Baseline Solver

Eigenvector 7

$> 10\times$ speedup

Multiscale Spectral Pb

Image

sPb

M-sPb Coarse

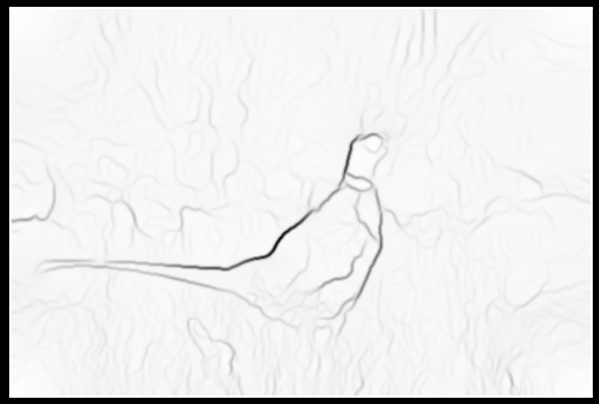
M-sPb Medium

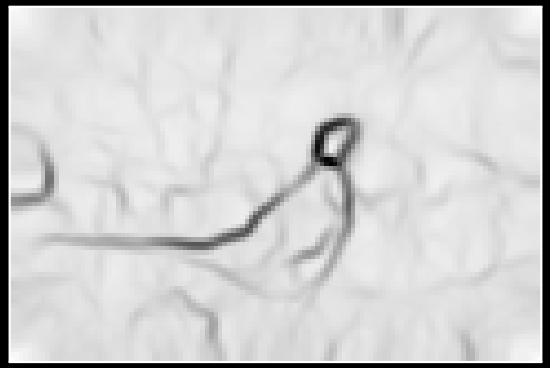
M-sPb Fine

Multiscale Spectral Pb

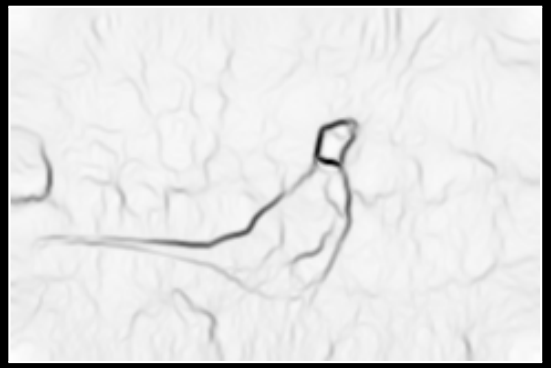
Image

sPb

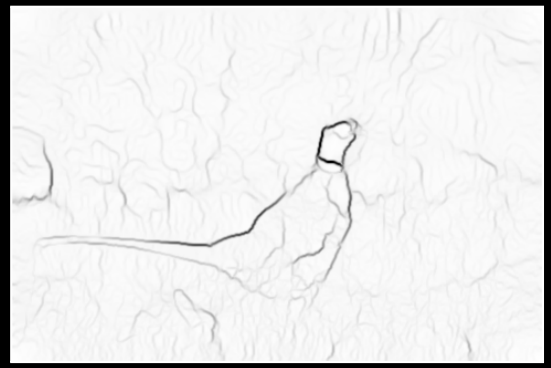




M-sPb Coarse



M-sPb Medium

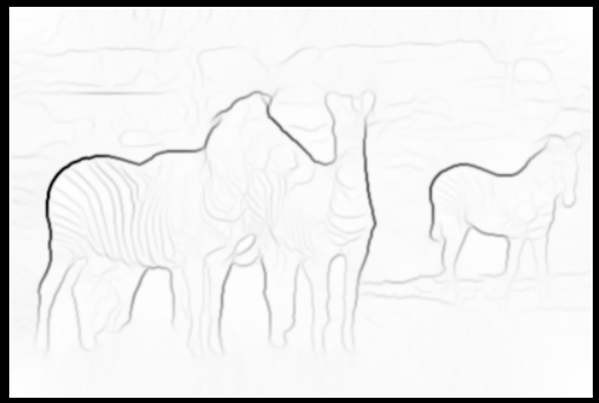


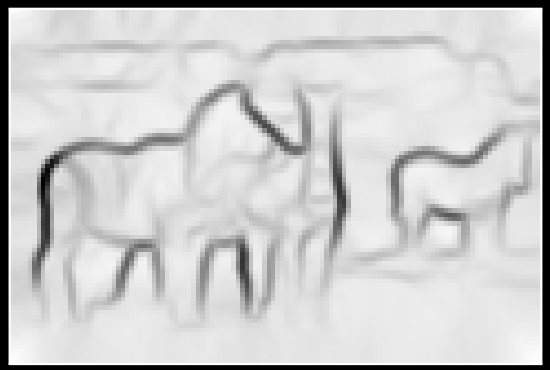
M-sPb Fine

Multiscale Spectral Pb

Image

sPb





M-sPb Coarse

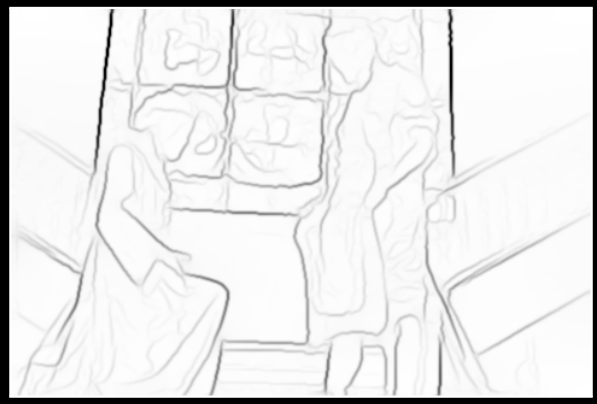
M-sPb Medium

M-sPb Fine

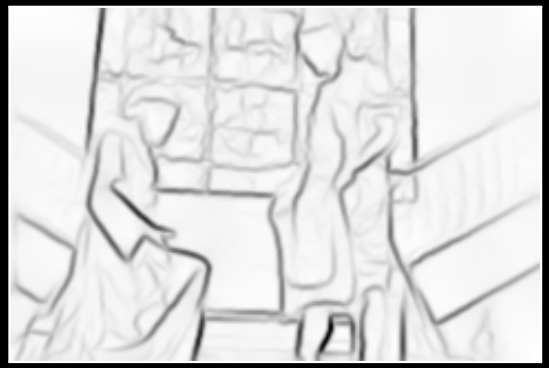
Multiscale Spectral Pb

Image

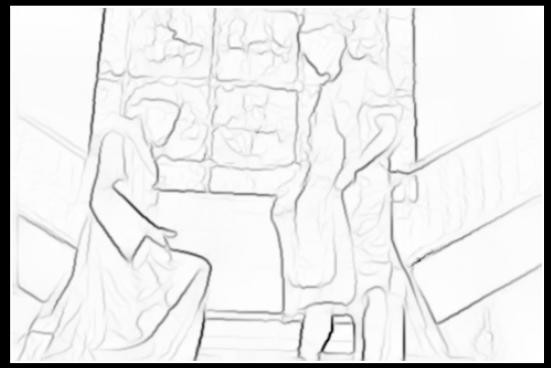
sPb



M-sPb Coarse



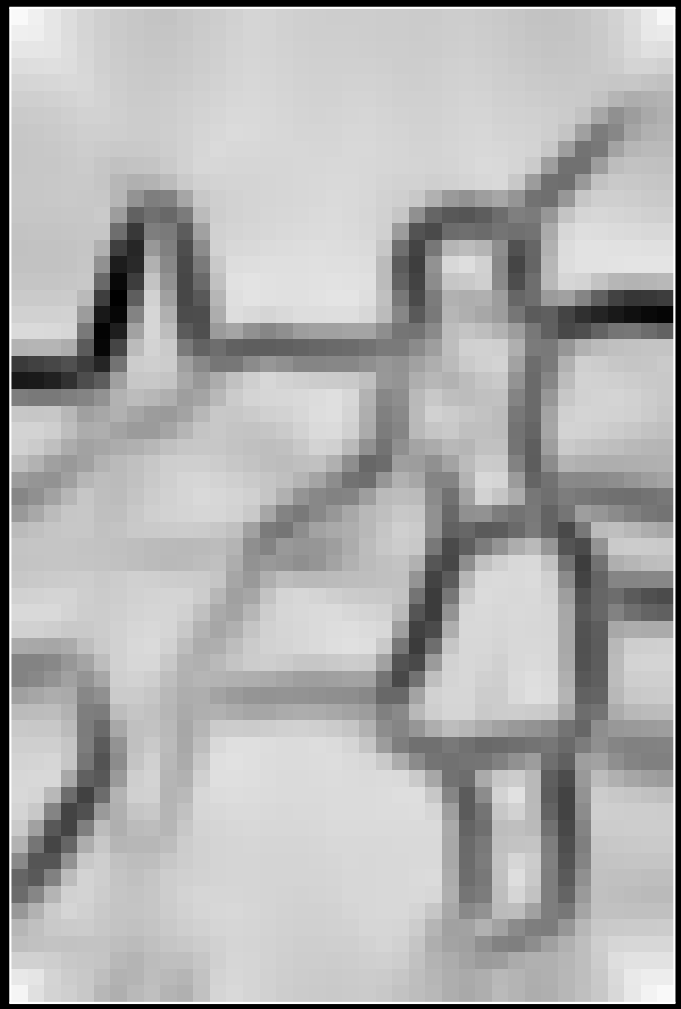
M-sPb Medium



M-sPb Fine

Multiscale Alignment for Free

Image



Multiscale sPb 1

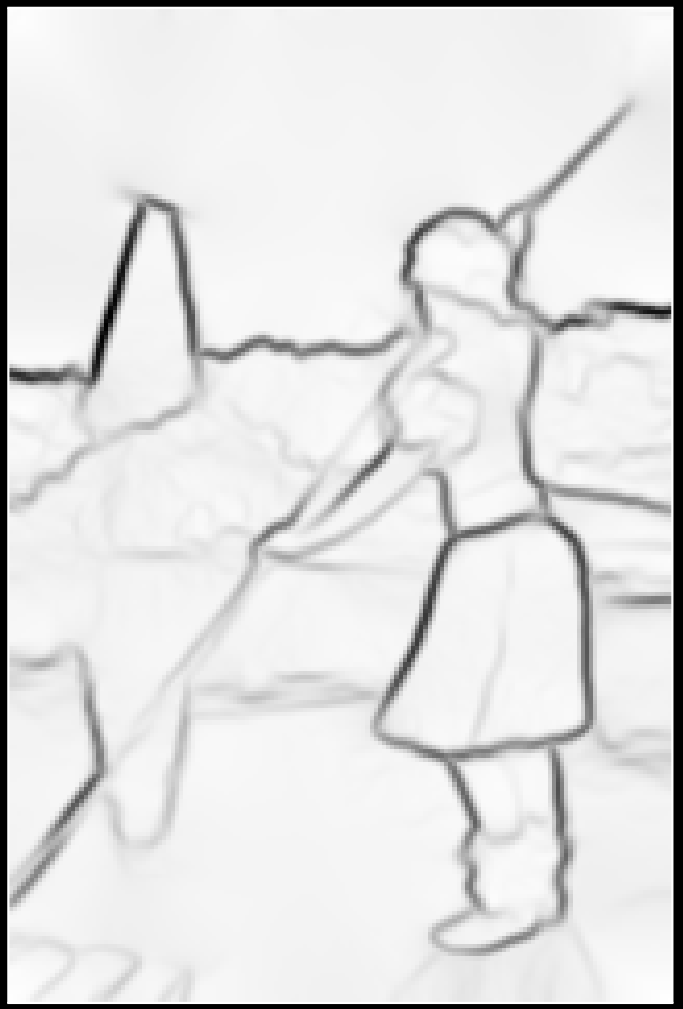
Multiscale Alignment for Free

Image

Multiscale sPb 2

Multiscale Alignment for Free

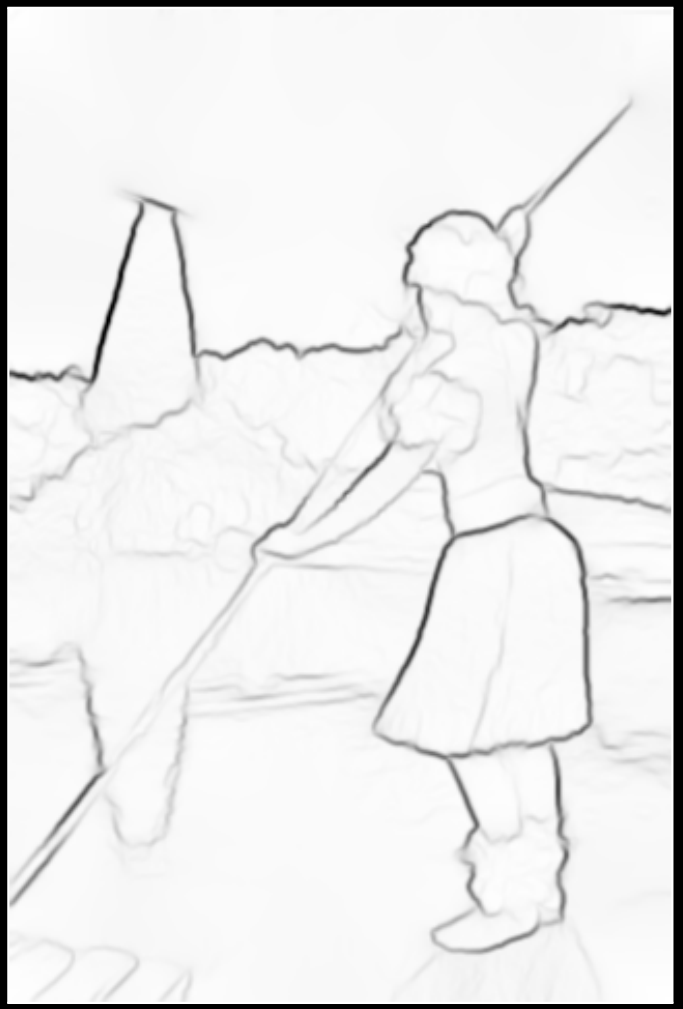
Image



Multiscale sPb 3

Multiscale Alignment for Free

Image



Multiscale sPb 4

Multiscale Alignment for Free

Image

Multiscale sPb 5

Thank You